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Connectedness and Classification of Certain Graphs*

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G. A. Dirac gives a necessary arc family condition for a graph to be n-vertex connected. The converse of this theorem of Dirac is false. Mesner and Watkins obtained partial results for additional conditions that the converse be true. A graph G which satisfies Dirac's arc family condition is now completely classified in terms of the order of $V(G)$, the structure of parts of minimum cutsets of G and consequent lower bounds for vertex-connectivity of G. Examples show that all lower bounds are best possible. Several distinct extensions of Whitney's necessary and sufficient condition for a graph to be n-vertex connected also appear as corollaries. Finally, examples are presented to show a graph which satisfies a given n-family arc condition. However, the same graph does not satisfy a very similar $(n-1)$-family arc condition where exactly one arc has been eliminated from the statement of the original n-family arc condition.

1. INTRODUCTION

In this paper, G denotes a finite undirected graph without loops and multiple edges. The degree of vertex-connectedness of G becomes of interest if it is known that a given configuration of an n-family of arcs exists between any two disjoint vertex subsets of G which contain given numbers of end points.

Whitney [8] gave a specific n-family arc condition having two singleton end points which is both necessary and sufficient for G to be n-vertex connected. Dirac [1] followed this by stating a generalized n-family arc condition which is necessary, but insufficient, for G to be n-vertex connected. Mesner and Watkins [4] began a study of additional conditions which together with Dirac's generalized condition would be sufficient for G to be n-vertex connected. This paper extends the investigation of Mesner and Watkins. A

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graph G which satisfies Dirac's generalized n-family arc condition is now classified with regard to structural features.

2. Preliminaries

The vertex set of G and the edge set of G are denoted, respectively, $V(G)$ and $E(G)$. The assertion that H is a subgraph of G is denoted $H \subseteq G$. If $U \subseteq V(G)$, then the *section subgraph* $G(U)$ has vertex set U; and edge $e \in E(G(U))$ if and only if $e \in E(G)$ and both end points of e belong to U.

In general, the terminology is that of Ore [5], but particular definitions are stated here for completeness. An *arc* is a sequence of distinct edges such that each pair of consecutive edges share a common vertex; and no vertex shall appear more than once except possibly as the initial and terminal end points of the arc. For disjoint X, $Y \subseteq V(G)$, an arc with one arc end point belonging to X and the other arc end point belonging to Y is said to be an *XY-arc*. If $X = \{x\}$, we write xY-arc rather than $\{x\}Y$-arc. An n-family of XY-arcs is said to be *openly disjoint* if the arcs are pairwise disjoint with respect to vertices and edges except possibly at common end points of arcs.

A graph G is *connected* if for any $x, y \in V(G)$ there exists an xy-arc. A graph G is *n-vertex connected* if $|V(G)| \geqslant n + 1$ and section subgraph $G(V(G) - Q)$ is connected for every $Q \subseteq V(G)$ such that $|Q| \leqslant n - 1$. The *vertex connectivity* of G, denoted $\kappa(G)$, is the maximal value of n for which G is n-vertex connected.

For a connected graph G, a subset $S \subseteq V(G)$ is called a *cutset* if section subgraph $G(V(G) - S)$ is not connected. For X, $Y \subseteq V(G)$, it is said that *cutset S separates X and Y* if X lies entirely within one component of $G(V(G) - S)$ and Y lies entirely within a second distinct component. A cutset having cardinality $\kappa(G)$ is a *minimum cutset*. The class of all minimum cutsets of G is denoted $C(G)$. Note that $C(G) = \varnothing$ if and only if G is a complete graph.

For $S \in C(G)$, a component P of $C(V(G) - S)$ is called *a part of G with respect to S*. Then it is said that *S admits part P*. Each vertex of a minimum cutset C must be adjacent to at least one vertex of every part admitted by C; otherwise, the minimality of C is contradicted. It follows that a part P admitted by minimum cutset S is uniquely associated with that $S \in C(G)$.

After Watkins [7, p. 24], we define

$$p(G) = \min\{\min\{|V(P)| : P \text{ is a part of } G \text{ with respect to } S\} : S \in C(G).\}.$$

Hence, $p(G)$ is the smallest number of vertices of any part of G. A part P is called an *atomic part* if $V(P) = p(G)$.

Let K_n denote the complete graph on n vertices and let $[V(G)]^q$ denote the Cartesian product of q copies of $V(G)$.

The following definition is an extension of a definition by Mesner and Watkins [4, p. 322]. We define the graph property which is denoted by the symbol $W_n(l_1,\ldots, l_q; k_1,\ldots, k_m)$: Let $|V(G)| \geqslant q+m$ and let $l_1,\ldots, l_q$ and $k_1,\ldots, k_m$ be two nonincreasing finite sequences of positive integers such that $\sum_{i=1}^{q} l_i = \sum_{j=1}^{m} k_j = n$. For any q-tuple of q distinct elements, $(a,\ldots, a_q) \in [V(G)]^q$, and for any m-tuple of m distinct elements, $(b_1,\ldots, b_m) \in [V(G)]^m$, such that $a_i \neq b_j$ for all i, j; there exists an openly disjoint n-family of AB-arcs, where $A = \{a_1,\ldots, a_q\}$ and $B = \{b_1,\ldots, b_m\}$. Furthermore, l_i of these n arcs are a_iB-arcs for $i = 1,\ldots, q$; and k_j of these n arcs are Ab_j-arcs for $j = 1,\ldots, m$.

It must be remarked immediately that if G satisfies $W(l_1,\ldots, l_q; k_1,\ldots, k_m)$, then this does not imply the existence of an a_ib_j-arc for any particular selection of i and j among A and B. The graph property merely requires that a_i be the end point of some l_i arcs among the n-family of arcs, but the second end point of these l_i arcs is not specified among B. A symmetric statement is true for b_j and k_j.

Given a particular pair of positive integer sequences $l_1,\ldots, l_q$ and $k_1,\ldots, k_m$ we denote the number of i for which $l_i = t$ by α_t, and we denote the number of j for which $k_j = t$ by β_t $(t = 1, 2, 3,\ldots)$. For example, consider $W_{21}(6, 4, 4, 3, 1, 1, 1, 1; 5, 3, 2, 2, 2, 2, 2, 2, 1)$. Then $\alpha_1 = 4$, $\alpha_2 = 0$, $\alpha_3 = 1$, $\alpha_4 = 2$, $\alpha_5 = 0$, $\alpha_6 = 1$, $\alpha_i = 0$ for $i \geqslant 7$; and $\beta_1 = 1$, $\beta_2 = 6$, $\beta_3 = 1$, $\beta_4 = 0$, $\beta_5 = 1$, $\beta_j = 0$ for $j \geqslant 6$.

If $l_i \geqslant 2$ (respectively $k_j \geqslant 2$), then we say a_i (respectively b_j), is a *multiple-arc end point* of the openly disjoint n-family of AB-arcs. Notice also,

$$\sum_{i=1}^{n} \alpha_i = q; \qquad \sum_{j=1}^{m} \beta_j = m; \qquad \text{as well as} \quad \sum_{i=1}^{q} l_i = \sum_{j=1}^{m} k_j = n.$$

Without loss of generality, the sequence of $l_1,\ldots, l_q$ may be interchanged with the sequence $k_1,\ldots, k_m$ due to the symmetric nature of the definition.

In terms of the above notation, the following important results have been previously obtained.

THEOREM 2.1 [8, Theorem 7]. *A necessary and sufficient condition that G be n-vertex connected is that G satisfy $W_n(n; n)$.*

In the above theorem, $q = m = 1$ and both A and B are singleton sets.

THEOREM 2.2 [1, Theorem B]. *Let G be a graph and let q, m, and n be positive integers such that $\kappa(G) \geqslant n$ and $q + m \leqslant |V(G)|$. Then G satisfies $W_n(l_1,\ldots, l_q; k_1,\ldots, k_m)$.*

In Theorem 2.2, it is necessary to restrict $q + m \leqslant |V(G)|$ in order for $(A \cup B) \subseteq V(G)$ and $A \cap B = \varnothing$. The converse of Theorem 2.2 is easily shown to be false by considering K_8. It is seen that K_8 satisfies $W_{16}(4, 4, 4, 4;$

4, 4, 4, 4) because any choice of $A, B \subseteq V(K_8)$ causes every vertex of A to be adjacent to all four vertices of B and vice versa. Each of the 16 required AB-arcs consists of a single edge. But $\kappa(K_8) = 7 < 16$.

Mesner and Watkins [4] began the study of additional conditions for the converse of Theorem 2.2 to be true by considering the following special case. Suppose that A is a singleton vertex and B contains at most one multiple-arc end point. Then $q = 1$, $l_1 = n$, $k_1 = n - m + 1$, and either $\beta_1 = m$ or $\beta_1 = m - 1$.

THEOREM 2.3 [4, Theorem 3]. *Let $m \leqslant n$ be positive integers. A necessary and sufficient condition that G be n-vertex connected is that G satisfy $W_n(n; n - m + 1, 1, 1,..., 1)$.*

For the arbitrary sequence $k_1, ..., k_m$, Mesner and Watkins then proved a somewhat more general result. Recall that $(m - \beta_1)$ is the number of multiple-arc end points of B.

THEOREM 2.4 [4, Theorem 4]. *Let n be a positive integer and let G satisfy $W_n(n; k_1, ..., k_m)$. If $\kappa(G) < n$, then*

$$|V(G)| \leqslant n - 3 + 2(m - \beta_1);$$
$$\text{and} \quad |V(G)| \leqslant 2n - 3, \text{ if } n \text{ is even};$$
$$\text{and} \quad |V(G)| \leqslant 2n - 4, \text{ if } n \text{ is odd}.$$

COROLLARY 2.4.1 [4, Corollary 4.1]. *Let $k_1, k_2, ..., k_m$ be such that $\sum_{j=1}^{m} k_j = n$. If either $|V(G)| \geqslant 2n - 2$ or $\beta_1 \geqslant m - 2$, then a necessary and sufficient condition that G satisfy $W_n(n; k_1, ..., k_m)$ is that G be n-vertex connected.*

The purpose of the following section is to extend the result of Theorem 2.4 to all $1 \leqslant |A| \leqslant n$ and $1 \leqslant |B| \leqslant n$ and to all sequences $l_1, ..., l_q$ and $k_1, ..., k_m$ where $\sum l_i = \sum k_j = n$. This leads to several corollaries which are extensions of Theorems 2.1 and 2.3. An extension of Corollary 2.4.1 is made to classify entirely a graph which satisfies $W_n(l_1, ..., l_q; k_1, ..., k_m)$.

3. OPENLY DISJOINT FAMILIES

The proofs of Theorems 2.3 and 2.4 pivot on the fact that $|A| = 1$. Thus $A = \{a\}$ can be chosen to belong to the vertex set of an atomic part P of G with respect to some $S \in C(G)$. Then the vertices of B can be chosen to have as many as possible contained in parts distinct from P which are admitted by the same $S \in C(G)$. This is followed by a counting argument to establish the minimum number of vertices in S which must exist as intermediate vertices of aB-arcs.

Several new definitions are useful to extend this counting argument to the general case of $W_n(l_1, ..., l_q; k_1, ..., k_m)$. Suppose $C \in C(G)$. The symbol $(V_1, V_2)_C$ denotes a *partition of the set of parts of G with respect to C.* Define $(V_1, V_2)_C$ to be an ordered pair of subsets of $V(G)$ such that (i) $V_1 \cap V_2 = \varnothing$ and $V_1 \cup V_2 = V(G) - C$; (ii) each section subgraph $G(V_i)$, $(i = 1, 2)$, is precisely a union of a subset of parts of G with respect to C; (iii) $(|V_2| - |V_1|) \geqslant 0$.

Clearly, $(V_1, V_2)_C$ need not be unique in G; for example, if C admits exactly three isomorphic parts of G. Furthermore, for any $(V_1, V_2)_C$,

$$|V_1| \leqslant \tfrac{1}{2}|V(G) - C| \leqslant |V_2|.$$

Suppose $(V_1, V_2)_S$ is found for $S \in C(G)$ and G satisfies $W_n(l_1, ..., l_q; k_1, ..., k_m)$ and it is possible to choose $A \subseteq V_1$ and $B \subseteq V_2$. It is immediate that S contains at least n elements which serve as intermediate vertices of the openly disjoint n-family of AB-arcs. This paper henceforth considers the cases where it is not possible for both $A \subseteq V_1$ and $B \subseteq V_2$ for any $(V_1, V_2)_S$.

Without loss of generality, we might assume $q \leqslant m$ and $|A| \leqslant |B|$ for $W_n(l_1, ..., l_q; k_1, ..., k_m)$. Then it is of interest whether any $S \in C(G)$ is such that $(V_1, V_2)_S$ has $|V_1| \geqslant |A|$. Accordingly, we define

$$h(G) = \max\{|V_1| : S \in C(G) \text{ and } V_1 \text{ is the first cell of } (V_1, V_2)_S\}.$$

An arbitrary $S \in C(G)$ need not admit $(V_1, V_2)_S$ such that $|V_1| = h(G)$. Since every part of G has at least $p(G)$ vertices, then $h(G) \geqslant p(G) \geqslant 1$. Note, also, that $h(K_m)$ is undefined since $C(K_n) = \varnothing$.

There is a very useful lower bound for $|V_i \cup S|$ of $(V_1, V_2)_S$, $(i = 1, 2)$, when G satisfies $W_n(l_1, ..., l_q; k_1, ..., k_m)$.

LEMMA 3.1. *Let $|V(G)| \geqslant q + m$ and let G satisfy $W_n(l_1, ..., l_q; k_1, ..., k_m)$. Let $S \in C(G)$. Then every $(V_1, V_2)_S$ satisfies both*

(i) $|V_i| + |S| \geqslant q + \max\{1, m - |V_j|\}$;

and

(ii) $|V_i| + |S| \geqslant m + \max\{1, q - |V_j|\}$;

for $i, j = 1, 2$; $i \neq j$.

Proof. Suppose $m \geqslant |V_i| + |S| = t$; $(i = 1, 2)$. Suppose that $b_1 \in V_1$ and $\{b_2, b_3, ..., b_t\} \subseteq (S \cup V_i)$. Then b_1 is adjacent only to other vertices of B. It follows that b_1A-arcs do not exclude all other vertices of B and this contradicts $W_n(l_1, ..., l_q; k_1, ..., k_m)$. Hence, $m \leqslant |V_i| + |S| - 1$, $(i = 1, 2)$. Similarly, $q \leqslant |V_i| + |S| - 1$, $(i = 1, 2)$. Since $|V_1| + |S| + |V_2| = |V(G)| \geqslant m + q$, then $|V_1| + |S| \geqslant m + q - |V_2|$, and $|V_2| + |S| \geqslant m + q - |V_1|$. The result now follows.

It is now possible to prove a theorem for the general case of $W_n(l_1, \ldots, l_q; k, \ldots, k_m)$ which extends Theorem 2.4. This theorem and corollaries establish that if G satisfies $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$, then G falls in exactly one of the following classes:

(i) G is n-vertex connected;

(ii) $\kappa(G) \leqslant n - 1$, and $|V(G)|$ is relatively not much greater than n;

(iii) $\kappa(G) \leqslant n - 1$, and every $S \in C(G)$ separates G into a single large part together with some small parts wherein the union of all small parts contains fewer vertices than either A or B.

Recall that α_1 is the number of elements of A which are end points of a single arc of the openly disjoint n-family of AB-arcs. Thus, $q - \alpha_1$ is the number of elements of A which are multiple-arc end points. Similarly, $m - \beta_1$ is the number of multiple-arc end points of B.

THEOREM 3.1. *Let* $|V(G)| \geqslant q + m$ *and let* G *satisfy* $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$.

(i) *If there exists* $S \in C(G)$ *and* $(V_1, V_2)_S$ *such that* $|V_1| \geqslant \min\{q - a_1, m - \beta_1\}$ *and* $|V_2| \geqslant \max\{q - \alpha_1, m - \beta_1\}$, *then* G *is* n*-vertex connected.*

(ii) *If* $h(G) \geqslant \min\{q - \alpha_1, m - \beta_1\}$, *then either* G *is* n*-vertex connected or* $|V(G)| \leqslant n + 2\max\{q - \alpha_1, m - \beta_1\} - 3$.

Proof. (i) Suppose $q - \alpha_1 \leqslant m - \beta_1$. Suppose $S \in C(G)$ and $(V_1, V_2)_S$ are such that $|V_1| \geqslant q - \alpha_1$ and $|V_2| \geqslant m - \beta_1$. Recall that $A' = \{a_1, a_2, \ldots, a_{q-\alpha_1}\}$ comprises the set of multiple-arc end points of A. This is because $l_1, \ldots, l_q$ is a nonincreasing sequence and there are $q - \alpha_1$ multiple-arc end points of A. Similarly, $B' = \{b_1, \ldots, b_{m-\beta_1}\}$ comprises the set of multiple-arc end points of B.

By supposition, we can choose $A' \subseteq V_1$ and $B' \subseteq V_2$. Next, we choose as many as possible of singleton-arc end points of $(A - A') = \{a_{q-\alpha_1+1}, \ldots, a_q\}$ to belong to V_1. By Lemma 3.1(i), the remainder of $(A - A')$ can be chosen to belong to S. Thus $A \subseteq (V_1 \cup S)$. Similarly, we choose as many as possible of $(B - B')$ to belong to V_2. By Lemma 3.1(ii), the remainder of $(B - B')$ can be chosen in S. Hence, $B \subseteq (V_2 \cup S)$. Altogether, S contains no multiple-arc end point of $(A \cup B)$.

Then each AB-arc with one arc end point belonging to $(V_1 \cap A)$ also contains a distinct element of $S - A$. This is because S contains no multiple-arc end points of B. Each arc with an end point belonging to $S \cap A$ contains a distinct element of S since S contains no multiple-arc end points of A. Hence, S contains at least one distinct element for each arc of the openly disjoint n-family of AB-arcs.

A symmetric result obtains if $m - \beta_1 \leqslant q - \alpha_1$, and the result follows.

(ii) Suppose $q - \alpha_1 \leqslant m - \beta_1$, so $h(G) \geqslant q - \alpha_1$ by hypothesis. Let $S \in (G)$ be such that $(V_1, V_2)_S$ has $|V_1| = h(G)$. If $|V_2| \geqslant (m - \beta_1)$, then G is n-vertex connected by Theorem 3.1(i). Suppose $|V_2| \leqslant (m - \beta_1) - 1$ and G is not n-vertex connected. Since $|V_1| \leqslant |V_2|$, it follows that $|V(G)| = |V(G)| = |V_1| + |S| + |V_2| \leqslant (m - \beta_1) - 1 + (n - 1) + (m - \beta_1) - 1 = n + 2(m - \beta_1) - 3$.

If $m - \beta_1 \leqslant q - \alpha_1$, then a similar argument shows that $|V(G)| \leqslant n + 2(q - \alpha_1) - 3$. The result now follows.

The first corollary includes the results of both Theorems 2.1 and 2.3. This corollary considers the case that A and B each contain at most one multiple-arc end point in possible conjunction with some singleton-arc end points. In this case, $\alpha_1 = q$ or $\alpha_1 = q - 1$ and $\beta_1 = m$ or $\beta_1 = m - 1$. Also, $l_2 = l_3 = \cdots = l_q = 1 = k_2 = k_3 = \cdots = k_m$ regardless of l_i and k_1. Notice that several corollaries consider all graphs whereas Theorem 3.1 considers only incomplete graphs.

COROLLARY 3.1.1. *Let $q \leqslant n$ and $m \leqslant n$ be positive integers such that $|V(G)| \geqslant q + m$. A necessary and sufficient condition that G be n-vertex connected is that G satisfy $W_n(n - q + 1, 1, 1,..., 1; n - m + 1, 1, 1,..., 1)$ such that $0 \leqslant q - \alpha_1 \leqslant 1$ and $0 \leqslant m - \beta_1 \leqslant 1$.*

Proof. Let G satisfy $W_n(n - q + 1, 1,..., 1; n - m + 1, 1,..., 1)$. Suppose $C(G) \neq \varnothing$. Any $S \in C(G)$ has $(V_1, V_2)_S$ such that $|V_2| \geqslant |V_1| \geqslant h(G) \geqslant p(G) \geqslant 1$. By hypothesis, $q - \alpha_1 \leqslant 1$ and $m - \beta_1 \leqslant 1$; so G is n-vertex connected by Theorem 3.1(i).

Suppose G is a complete graph. If $n - q + 1 \leqslant m$, then $n + 1 \leqslant q + m \leqslant |V(G)|$ and complete graph G is at least n-vertex connected. If $l_1 = n - q + 1 \geqslant m$, then a_1 may have at most m-single edge a_1B-arcs. The remaining $(n - q + 1) - m$ arcs from a_1 to B must each contain at least one distinct vertex which serves as an intermediate of an a_1B-arc which consists of at least two edges. It follows that $|V(G)| \geqslant q + m + (n - q + 1) - m = n + 1$, so complete G is at least n-vertex connected.

Let G be n-vertex connected. Then G satisfies $W_n(n - q + 1, 1,..., 1; n - m + 1, 1,..., 1)$ by Theorem 2.2.

In the case, $q = n = m$, Corollary 3.1.1 includes an independent proof of a result previously reported by Harary [3].

The next corollary assumes a lower bound for $V(G)$ and also for $h(G)$ where $h(G)$ is defined. Then there arises a necessary and sufficient condition that G be n-vertex connected.

COROLLARY 3.1.2. (i). *Let G be such that $|V(G)| \geqslant q + m$. Let $W_n(l_1,..., l_q; k_1,..., k_m)$ be such that $h(G) \geqslant \min\{q - \alpha_1, m - \beta_1\}$ and $|V(G)| \geqslant$*

$n + 2\max\{q - \alpha_1, m - \beta_1\} - 2$. Then a necessary and sufficient condition that G be n-vertex connected is that G satisfy $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$.

(ii) *Let complete G be such that $|V(G)| \geqslant q + m$. Let $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$ be such that $|V(G)| \geqslant n + 2\max\{q - \alpha_1, m - \beta_1\} - 2$. Then a necessary and sufficient condition that G be n-vertex connected is that G satisfy $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$.*

Proof. (i) Let G be n-vertex connected. Then G satisfies $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$ by Theorem 2.2.

Let G satisfy $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$. Suppose G is not complete. By hypothesis, $h(G) \geqslant \min\{q - \alpha_1, m - \beta_1\}$ and $|V(G)| \geqslant n + 2\max\{q - \alpha_1, m - \beta_1\} - 2$. It follows from Theorem 3.1(ii) that G is n-vertex connected.

(ii) If G is n-vertex connected, then G satisfies $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$ by Theorem 2.2.

If $\max\{q - \alpha_1, m - \beta_1\} \leqslant 1$, then G is n-vertex connected by Corollary 3.1.1. If $\max\{q - \alpha_1, m - \beta_1\} \geqslant 2$, then $|V(G)| \geqslant n + 2(2) - 2 = n + 2$, and complete G is at least $(n + 1)$-vertex connected.

The following corollary considers the case where one set of end points of the openly disjoint AB-arc family contains at most one multiple-arc end point and the other set of end points contains exactly two multiple-arc end points.

COROLLARY 3.1.3. *Let G be such that $|V(G)| \geqslant q + m$. Let $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$ be such that $q - 1 \leqslant \alpha_1 \leqslant q$ and $\beta_1 = m - 2$. Then a necessary and sufficient condition for G to be n-vertex connected is that G satisfy $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$.*

Proof. Let G be n-vertex connected. Then G satisfies $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$ by Theorem 2.2.

Let G satisfy $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$. If there exists $S \in C(G)$ such that $(V_1, V_2)_S$ has $|V_1| \geqslant 1$ and $|V_2| \geqslant 2$, then G is n-vertex connected by Theorem 3.1(i). Suppose every $S \in C(G)$ has $(V_1, V_2)_S$ with $|V_1| = |V_2| = 1$. Choose $a_1 \in V_1$ and $b_1 \in V_2$ and $b_2 \in S$ for some $(V_1, V_2)_S$. By hypothesis $q - 1 \leqslant \alpha_1 \leqslant q$, so every element of $A \cap S$ is a singleton-arc end point of A. Since G is simple, there can exist at most one single-edge a_1b_2-arc. It now follows that b_1 is adjacent to at least k_1 distinct elements of $S - B$ and b_2 is adjacent to at least $(k_2 - 1)$ distinct elements of $S - B$ in order for the openly disjoint n-arc family to exist. In addition, $\{b_2, \ldots, b_m\} \subseteq S$. Hence, $|S| \geqslant k_1 + (k_2 - 1) + (m - 1) = k_1 + k_2 + (m - 2) = k_1 + k_2 + \beta_1 = n$, since $m - 2 = \beta_1$ by hypothesis.

Suppose G is a complete graph. There exists at most one single-edge a_1b_1-arc and at most one single-edge a_1b_2-arc. Since a_1 is the only possible multiple-arc end point of A, then b_1 must be adjacent to $(k_1 - 1)$ vertices distinct from $(a_1 \cup B)$. Also, b_2 must be adjacent to $(k_2 - 1)$ vertices distinct from $(a_1 \cup B)$. Hence, $|V(G)| \geqslant |a_1 \cup B| + (k_1 - 1) + (k_2 - 1) = (1 + m) + k_1 + k_2 - 2 = 1 + \beta_1 + k_1 + k_2 = 1 + n$. Thus complete graph G is at least n-vertex connected.

Together Corollaries 3.1.2 and 3.1.3 include Corollary 2.4.1.

4. Lower Bound of $\kappa(G)$

Corollaries 3.2.1 to 3.1.3 state necessary and sufficient conditions that any graph to be n-vertex connected. Concise proofs are given for these particular cases. However, the sufficiency argument for incomplete graphs may be extended to provide a generalized lower bound for vertex-connectivity.

Suppose G satisfies $W_n(l_1, ..., l_q; k_1, ..., k_m)$ and $\kappa(G) \leqslant n - 1$. By Theorem 3.1(i), every $(V_1, V_2)_S$ is such that it is impossible for all multiple-arc end points of A to be contained in V_i and all multiple-arc end points of B to be contained in V_j, $i \neq j$. Thus, some part P with respect to S is such that $S \cup V(P)$ contains a multiple-arc end point of one set together with at least one element of the opposite set of arc end points. Hence, there is collapsing of the necessary number of AB-arcs which pass through S from V_1 to V_2. It is no longer true that every AB-arc contains a distinct element of S. In order to generalize the lower bound for $\kappa(G)$ then we count the reduction in necessary number of elements of S due to:

(i) single-edge AB-arcs between multiple-arc end points of A and multiple-arc end points of B within section subgraph $G(S)$;

(ii) multiple-arc end points in S which have arcs running directly to adjacent vertices which are not elements of S.

Consider a nonincreasing sequence of positive integers $k_1, ..., k_m$ such that $\sum_{j=1}^{m} k_j = n$. As usual, let β_t be the number of j for which $k_j = t$. Then

$$\beta_j = 0 \quad \text{for } j \geqslant n + 1; \qquad \sum_{j=1}^{n} j\beta_j = n; \qquad \sum_{j=1}^{n} \beta_j = m.$$

Let N_2 correspond to $|V_i|$ for V_i belonging to some $(V_1, V_2)_S$. Suppose $N_2 \leqslant m - 1$. Consider that $f(u) = \sum_{j=u}^{n} \beta_j$ is a nonincreasing function in u ranging from $f(1) = m$ to $f(n) = \beta_n \leqslant 1$. Hence, it is always possible to find a minimal positive integer s such that $f(s) = \sum_{j=s}^{n} \beta_j \leqslant N_2$. By the minimality of s, it follows that $N_2 \leqslant f(s - 1) - 1$. For such values of s, we define nonnegative integer $w = N_2 - \sum_{j=s}^{n} \beta_j$. Then $0 \leqslant w \leqslant \beta_{s-1} - 1$. By definition, s is the smallest positive integer such that we can choose $b_j \in V_i$ for every $k_j \geqslant s$. It follows that w is then the maximum number of b_j which can be chosen in V_i where $k_j = s - 1$. All b, with $k_j \leqslant s - 2$ must be chosen outside V_i. In the case $N_2 \geqslant m$, define $w = 0$ and $s = 1$. Finally, define $M = N_2 - m$ if $N_2 \geqslant m$ and define $M = 0$ if $N_2 \leqslant m - 1$.

Similarly, we define r, t, Q for a nonincreasing sequence of positive integers $l_1, ..., l_q$ and positive integer N_1. Let N_1 correspond to $| V_i |$ for V_i belonging to $(V_1, V_2)_S$. Suppose $N_1 \leqslant q - 1$. Consider $g(u) = \sum_{j=u}^{n} \alpha_j$ a nonincreasing function in u ranging from $g(1) = q$ to $g(n) = \alpha_n \leqslant 1$. Hence, it is always possible to find a minimal positive integer r such that $g(r) \leqslant N_1 \leqslant g(r-1) - 1$.

For such value of r, define $t = N_1 - \sum_{j=r}^{n} \alpha_j$. In the case $N_1 \geqslant q$, define $r = 1$ and $t = 0$. Let $Q = N_1 - q$ in the case $N_1 \geqslant q$ and let $Q = 0$ when $N_1 \leqslant q - 1$.

These rather inelegant expressions for N_1 and N_2 are admitted in order to state a single unified theorem rather than many distinct but similar cases. To summarize,

$$N_1 = \sum_{j=r}^{n} \alpha_j + t + Q \qquad \text{and} \qquad N_2 = \sum_{j=s}^{n} \beta_j + w + M.$$

THEOREM 4.1. *Let G satisfy $W_n(l_1, ..., l_q; k_1, ..., k_m)$ such that $| V(G) | \geqslant q + m$. Suppose $S \in C(G)$ has $V_1, V_2 \subseteq V(G)$ such that V_1, V_2, and S are pairwise disjoint and $V(G) = V_1 \cup S \cup V_2$ while no vertex of V_1 is adjacent to any vertex of V_2. Suppose for $c, d = 1, 2 (c \neq d)$ that*

$$| V_c | = \sum_{j=r}^{n} \alpha_j + t + Q \qquad \textit{and} \qquad | V_d | = \sum_{j=s}^{n} \beta_j + w + M,$$

for r, t, s, w, Q, M as defined above. Then

$$\kappa(G) \geqslant n - \sum_{j=1}^{r-1} (j-1)\, \alpha_j - \sum_{j=1}^{s-1} (j-1)\, \beta_j + tr + ws - 2(t + w). \qquad \text{(I)}$$

Remark. The proof technique is an extension of the proof of Theorem 3.1. If $s \leqslant 2$ and $r \leqslant 2$, then $\kappa(G) \geqslant n$ by (I) above. This is precisely the statement of Theorem 3.1(i). Note that as r and s increase, the number of multiple-arc end points found in S must also increase; and the lower bound of (I) must rapidly decrease. Examples will be given to show that the bound of (I) is best possible.

Proof. Let $S \in C(G)$ be as in the hypothesis so that $| V_c | \geqslant y$ and $| V_d | \geqslant z$ for $y = \sum_{j=r}^{n} \alpha_j + t$ and $z = \sum_{j=s}^{n} \beta_j + w$. Choose $a_1, ..., a_y \in V_c$ and choose $b_1, ..., b_z \in V_d$. If $y \leqslant q - 1$ or $z \leqslant m - 1$, then choose $a_{y+1}, ..., a_q \in S$ and $b_{z+1}, ..., b_m \in S$ by Lemma 3.1.

There are $\sum_{j=r}^{n} \alpha_j$ multiple-arc end points of $(A \cap V_c)$ which each serve as the end point of at least r of the AB-arcs. There are t multiple-arc end points of $(A \cap V_c)$ which each serve as the end point of exactly $(r - 1)$ of the AB-arcs. There are $\sum_{j=r}^{n} \beta_j$ multiple-arc end points of $(B \cap V_d)$ which each serve as the end point of at least s of the AB-arcs. There are w multiple-arc end points of $(B \cap V_d)$ which each serve as the end point of exactly $(s - 1)$ of the AB-arcs. It follows that there are exactly $\sum_{j=1}^{s-1} j\beta_j - w(s - 1)$ of the AB-arcs which have one end point belonging to $(B \cap S)$.

The openly disjoint n-family of AB-arcs has exactly $\sum_{j=r}^{n} j\alpha_j + t(r - 1)$ arcs with one end point belonging to $\{a_1 ,..., a_y\} \subseteq V_c$. Among this subset of AB-arcs, at most $\sum_{j=1}^{s-1} j\beta_j - w(s - 1)$ of these AB-arcs have one end point belonging to $B \cap S$. Hence, at least $\sum_{j=r}^{n} j\alpha_j + t(r - 1) - \sum_{j=1}^{s-1} j\beta_j - w(s - 1)$ of these arcs then contain a distinct vertex of $S - (A \cup B)$ as an intermediate vertex of a $(A \cap V_c)$-$(B \cap V_d)$-arc. It follows that

$$\begin{aligned}
\kappa(G) &= |S| \\
&\geqslant \sum_{j=r}^{n} j\alpha_j + t(r-1) - \left(\sum_{j=1}^{s-1} j\beta_j - w(s-1)\right) + |S \cap (A \cup B)| \\
&= n - \sum_{j=1}^{r-1} j\alpha_j + t(r-1) \\
&\quad - \left(\sum_{j=1}^{s-1} j\beta_j - w(s-1)\right) + \sum_{j=1}^{r-1} \alpha_j - t + \sum_{j=1}^{s-1} \beta_j - w \\
&= n - \sum_{j=1}^{r-1} (j-1)\, \alpha_j - \sum_{j=1}^{s-1} (j-1)\, \beta_j + tr + ws - 2(t + w).
\end{aligned}$$

In order to present examples, we define a certain class of graphs. Let $\theta_1 , \theta_2 , \theta_3$ be any three positive integers and let $G(\theta_1 , \theta_2 , \theta_3)$ denote a graph G as follows:

Let $V(G) = U_1 \cup U_2 \cup U_3$; $| U_i | = \theta_i , (i = 1, 2, 3)$; and U_1 , U_2 , U_3 are pairwise disjoint. Form $G(\theta_1 , \theta_2 , \theta_3)$ from the complete graph on $U_1 \cup U_2 \cup U_3$ by removing every edge of the complete graph which has one end point in U_1 and the other end point in U_3.

Note that U_2 is the only minimum cutset of $G(\theta_1 , \theta_2 , \theta_3)$ since section subgraphs $G(U_1 \cup U_2)$ and $G(U_2 \cup U_3)$ are each complete. Hence, $\kappa(G(\theta_1 , \theta_2 , \theta_3)) = \theta_2$.

We now consider infinite families of graphs which demonstrate that the lower bound of (I) is best possible for $\kappa(G) \leqslant n - 1$. Many types of examples are available, but only two of the more significant, yet contrasting, types are included here. Further examples are quite similar in construction. Both examples demonstrate equality in (I) for $\kappa(G) = n - 1$ while G satisfies $W_n(l_1 ,..., l_q ; k_1 ,..., k_m)$ for arbitrarily large n.

By way of contrast:

(i) Example 4.1 has $|V_1| = |V_2| = q - \alpha_1 - 1 = m - \beta_1 - 1$. Thus, each cell of the partition never contains all the multiple-arc endpoints of either A or B.

(ii) Example 4.2 has $|V_1| = q - \alpha_1 - 1 = m - \beta_1 - 1$ and $|V_2| = q - \alpha_1 = m - \beta_1$. Thus, V_2 may contain all the multiple-arc end points of say A, while V_1 cannot contain all the multiple-arc end points of B.

This particular pair of examples will also serve in Sections 5 and 6. Other obvious types of examples would include examples such as $\min\{q - \alpha_1, m - \beta_1\} \leqslant |V_1| \leqslant |V_2| \leqslant \max\{q - \alpha_1, m - \beta_1\}$ and $|V_1| < |V_2| < \min\{q - \alpha_1, m - \beta_1\}$. The argument that G satisfies $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$ of the following examples may be found in the reference. The details of these arguments are straightforward and lengthy and are omitted here.

EXAMPLE 4.1(i) [2, Example 2.1(i)].

n = 2k (k ⩾ 3)

$G(k-1, 2k-2, k-1)$ satisfies $W_{2k}(2, \ldots, 2; 2, \ldots, 2)$.

$q = m = k = \alpha_2 = \beta_2 \qquad \kappa(G) = 2k - 2$

$\alpha_i = \beta_i = 0; (i \neq 2) \qquad h(G) = p(G) = k - 1$

$q - \alpha_1 = m - \beta_1 = k = h(G) + 1$

The unique $S \in C(G)$ has $|S| = 2k - 2$ and $(V_1, V_2)_S$ has $|V_1| = |V_2| = k - 1$. Furthermore, $h(G) = \sum_{j=3}^{n} \alpha_j + k - 1 = \sum_{j=3}^{n} \beta_j + k - 1$. Thus, for application of (I) we have $r = s = 3$ and $t = w = k - 1$. By Theorem 4.1,

$$\begin{aligned}\kappa(G) &\geqslant n - \sum_{j=1}^{r-1} (j-1)\,\alpha_j - \sum_{j=1}^{s-1} (j-1)\,\beta_j + tr + ws - 2(t + w)\\ &= 2k - (k) - (k) + 3(k-1) + 3(k-1) - 2(2k-2)\\ &= 2k - 2, \text{ which is the actual vertex connectivity of } G.\end{aligned}$$

EXAMPLE 4.1 (ii) [1, Example 2.1(ii)].

n = 2k + 1 (k ⩾ 3)

$G(k-1, 2k-1, k-1)$ satisfies $W_{2k+1}(3, 2, \ldots, 2; 3, 2, \ldots, 2)$

$q = m = k \qquad \kappa(G) = 2k - 1$

$\alpha_2 = \beta_2 = k - 1 \qquad h(G) = p(G) = k - 1$

$\alpha_3 = \beta_3 = 1$

$\alpha_i = \beta_i = 0; (i \neq 2, 3)$

$q - \alpha_1 = m - \beta_1 = k = h(G) + 1$
$r = s = 3$ and $t = w = k - 2$

By (I) of Theorem 4.1;

$\kappa(G) \geqslant (2k + 1) - (k - 1) - (k - 1) + 3(k - 2) + 3(k - 2) - 2(k - 4)$
$= 2k - 1$; the actual vertex connectivity of G.

EXAMPLE 4.1(iii) [1, Example 2.1(iii)].

n = 5

$G(1, 4, 2)$ satisfies $W_5(3, 2; 3, 2)$ where $\kappa(G) = 4$.
By (I) of Theorem 4.1, $\kappa(G) \geqslant 4$.

n = 4 $G(1, 3, 2)$ satisfies $W_4(2, 2; 2, 2)$.

By (I) of Theorem 4.1, $\kappa(G) \geqslant 3$.
If $n \leqslant 3$, then $\kappa(G) \geqslant 3$, by Corollary 3.1.1. This follows since at most one element of either A or B may be a multiple-arc end point.

Thus, Example 4.1 includes a graph G for each $n \geqslant 4$ wherein G satisfies some condition for an n-family of openly disjoint arcs and $\kappa(G) = n - 1$, the lower bound (I) of Theorem 4.1.

EXAMPLE 4.2(i) [1, Example 2.2(i)].

n = 2k (k ⩾ 3)

$G(k - 1, 2k - 1, k)$ satisfies $W_{2k}(2, \ldots, 2; 2, \ldots, 2)$
$q = m = k = \alpha_2 = \beta_2 \qquad \kappa(G) = 2k - 1$
$\alpha_i = \beta_i = 0; (i \neq 2) \qquad h(G) = p(G) = k - 1$
$q - \alpha_1 = m - \beta_1 = k$
$r = 3; t = k - 1; s = 1; w = 0$

By (I) of Theorem 4.1, $\kappa(G) \geqslant 2k - 1$, the actual vertex-connectivity of G.

EXAMPLE 4.2(ii) [1, Example 2.2(i)].

n = 2k + 1 (k ⩾ 3)

$G(k - 1, 2k, k)$ satisfies $W_{2k+1}(3, 2, \ldots, 2; 3, 2, \ldots, 2)$
$q = m = k \qquad \kappa(G) = 2k$
$\alpha_2 = \beta_2 = k - 1 \qquad h(G) = p(G) = k - 1$

$\alpha_3 = \beta_3 = 1$

$\alpha_i = \beta_i = 0; \ (i \neq 2, 3)$

$q - \alpha_1 = m - \beta_1 = k$

By (I) of Theorem 4.1, $\kappa(G) \geqslant 2k$, the actual vertex-connectivity of G.

Thus, both Examples 4.1 and 4.2 show that the lower bound of (I) is best possible. The following Theorem 4.2 may be proven as a corollary of Theorem 4.1, but a direct proof is more efficient. The result of Theorem 4.2 is more specific than Theorem 4.1 and this result is useful in Section 5.

THEOREM 4.2. *Let G satisfy $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$ and let* $|V(G)| \geqslant q + m$.

(i) *If* $n = 4, 5$, *then* $\kappa(G) \geqslant n - 1$.

(ii) *If* $n = 1, 2, 3$, *then* $\kappa(G) = n$.

Proof. (i) Suppose G satisfies $W_5(l_1, \ldots, l_q; k_1, \ldots, k_m)$; it is impossible that $\max\{q - \alpha_1, m - \beta_1\} \geqslant 3$ for $n = 5$. If either A or B has fewer than two multiple-arc end points, then $\kappa(G) \geqslant 5$ by Corollary 3.1.1 or Corollary 3.1.3. Suppose A and B each have exactly two multiple-arc end points. If $S \in C(G)$ has $(V_1, V_2)_S$ with $|V_1| \geqslant 2$ then $\kappa(G) \geqslant 5$ by Theorem 3.1.

Suppose $|V_1| = 1$. If $l_1 = 3$, then we choose $a_1 \in V_1$ and $a_2 \in S$. It follows that at least three elements of $S - \{a_2\}$ must be adjacent to a_1. Hence, $\kappa(G) = |S| \geqslant 4$. If G is complete, then $|V(G)| \geqslant 5$ by a similar argument. Hence $\kappa(G) \geqslant 4$. If $l_1 = l_2 = 2$, $l_3 = 1$, then we choose $a_1 \in V_1$, $a_2 \in S$, $a_3 \in S$. Since a_1 must be adjacent to at least two vertices of $S - \{a_2, a_3\}$, then $\kappa(G) = |S| \geqslant 4$ for incomplete G. Again, $|V(G)| \geqslant 5$ for a complete graph.

Suppose G satisfies $W_4(l_1, \ldots, l_q; k_1, \ldots, k_m)$. It follows that $\kappa(G) \geqslant 3$ by an argument which is strictly analagous to the previous case of $n = 5$.

(ii) If G satisfies $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$ for $n = 1, 2, 3$, then $\max\{q - \alpha_1, m - \beta_1\} \leqslant 1$ and G is n-vertex connected by Corollary 3.1.1.

5. REDUCTION OF ARC FAMILY

Suppose G is p-vertex connected. By Theorem 2.2, then G satisfies any $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$ for which $n \leqslant p$ and $|V(G)| \leqslant q + m$. For example, consider complete graph K_8 wherein $\kappa(K_8) = 7$. Then K_8 satisfies $W_6(6; 4, 2)$ and $W_4(2, 2; 1, 1, 1, 1)$ and $W_7(7, 7)$.

It is somewhat surprising to discover a certain large class of graphs wherein each G satisfies some $W_n(l_1, \ldots, l_q; k_1, \ldots, k_m)$ and $\kappa(G) \leqslant n - 1$. In each of these graphs the reduction of one arc from the n-family arc property then prohibits G from satisfying the modified property for an openly disjoint $(n-1)$-family of arcs. More precisely, G satisfies $W_n(l_1,..., l_q; k_1,..., k_m)$, but the same G does not satisfy $W_{n-1}(l_1,..., l_{q-1}, l_q - 1; k_1,..., k_{m-1}, k_m - 1)$ for identical n, l_i, k_j values.

For example, an elementary though tedious check will convince the reader that $G(2, 4, 2)$ satisfies $W_6(2, 2, 2; 2, 2, 2)$. It is easily seen, as follows, that G does not satisfy $W_5(2, 2, 1; 2, 2, 1)$. Choose $a_1, a_2 \in U_1$; $a_3, b_3 \in U_2$; $b_1, b_2 \in U_3$. A pair of single edges serve for two $\{a_1, a_2\}$ b_3-arcs. There are still three $\{a_1, a_2\}$ $\{b_1, b_2\}$-arcs required, but only two elements of U_2-$(A \cup B)$ are available to serve as intermediate vertices of the openly disjoint 5-arc family. Hence, G does not satisfy $W_5(2, 2, 1; 2, 2, 1)$.

This phenomenon is due to the following: As G satisfies $W_n(l_1,..., l_q; k_1,..., k_m)$, at least one multiple-arc end point of A and at least one multiple-arc end point of B must both belong to the same minimum cutset C of G. After a reduction in the arc family property there are fewer multiple-arc end points which are forced to belong to C. This in turn requires more arcs to pass through C between end points in distinct parts with respect to C. However, there are not sufficient elements of C to serve as intermediate vertices of arcs between parts.

A more general example is proved by Examples 4.1(i) and (ii). It was shown that $G(t-1, 2t-2, t-1)$ satisfies $W_{2t}(2,..., 2; 2,..., 2)$; where $n = 2t$, $q = m = t \geqslant 3$, and $\alpha_2 = t$. However, it is shown [2, p. 38] that $G(t-1, 2t-2, t-1)$ does not satisfy $W_{2t-1}(2,..., 2, 1; 2,..., 2, 1)$; where $n = 2t-1$, $q = m = t \geqslant 3$, $\alpha_1 = \beta_1 = 1$, and $\alpha_2 = \beta_2 = t-1$. This latter argument proceeds the same as for $G(2, 4, 2)$ and $W_5(2, 2, 1; 2, 2, 1)$. Similarly, $G(t-1, 2t-1, t-1)$ satisfies $W_{2t+1}(3, 2,..., 2; 3, 2,..., 2)$; but $G(t-1, 2t-2, t-1)$ does not satisfy W_2 $(3, 2,..., 2, 1; 3, 2,..., 2, 1)$ where $n = 2t$, $q = m = t \geqslant 3$, $\alpha_3 = \beta_3 = \alpha_1 = \beta_1 = 1$, $\alpha_2 = \beta_2 = t-2$.

We now summarize the previous discussion together with the result of Theorem 4.2 in a single theorem.

THEOREM 5.1. (i) *Let $p = 1, 2, 3$. Let G satisfy $W_p(l_1',..., l_q'; k_1',..., k_{m'}')$ and $|V(G)| \geqslant q' + m'$. For any positive integers n, q, m such that $n \leqslant p$, $q + m \leqslant q' + m'$, then G also satisfies $W_n(l_1,..., l_q; k_1,..., k_m)$.*

(ii) *Let $p = 4, 5$. Let G satisfy $W_p(l_1',..., l_{q'}; k_1',..., k_{m'}')$ and $|V(G)| \geqslant q' + m'$. For any positive integers n, q, m such that $n \leqslant p-1$ and $q + m \leqslant q' + m'$ then G satisfies $W_n(l_1,..., l_q; k_1,..., k_m)$.*

(iii) *Let $p \geqslant 6$. Then there exists some G which satisfies some W_p $(l_1',..., l_{q'}; k_1',..., k_{m'}')$, but G does not satisfy some other $W_n(l_1,..., l_q; k_1,..., k_m)$ wherein $n \leqslant p-1$, and $q + m \leqslant q' + m'$.*

Proof. (i)(ii) Immediate from Theorems 4.2 and 2.2. (iii) Such examples are given in Example 4.1.

The reduction of a single arc is not essential to the phenomenon of Theorem 5.1(iii). It is possible to eliminate all the arcs of a multiple-arc end point.

EXAMPLE 5.1 [1, Example 2.3]. (i) **n = 3t**, $(t \geqslant 4)$; $G(t-1, 3t-4, t-1)$ satisfies $W_{3t}(3,..., 3; 3,..., 3)$ wherein $\alpha_3 = \beta_3 = q = m = t$. But $G(t-1, 3t-4, t-1)$ does not satisfy $W_{3t-3}(3,..., 3; 3,..., 3)$ wherein $\alpha_3 = \beta_3 = q = m = t-1$.

(ii) **n = 3t + 1**, $(t \geqslant 4)$; $G(t-1, 3t-3, t-1)$ satisfies $W_{3t+1}(4, 3,..., 3; 4, 3,..., 3)$ wherein $\alpha_3 = \beta_3 = q-1 = m-1 = t-1$. But $G(t-1, 3t-3, t-1)$ does not satisfy $W_{3t-2}(4, 3,..., 3; 4, 3,..., 3)$ wherein $\alpha_3 = \beta_3 = t-2$.

(iii) **n = 3t + 2**, $(t \geqslant 4)$; $G(t-1, 3t-2, t-1)$ satisfies $W_{3t+2}(5, 3,..., 3; 5, 3,..., 3)$ wherein $\alpha_3 = \beta_3 = q-1 = k-1 = t-1$. But $G(t-1, 3t-2, t-1)$ does not satisfy $W_{3t-1}(5, 3,..., 3; 5, 3,..., 3)$ wherein $\alpha_3 = \beta_3 = t-2$.

Further examples follow easily from similar techniques.

6. CLASSIFICATION

It is now possible to classify a graph G which satisfies $W_n(l_1,..., l_q; k_1,..., k_m)$. Examples show that bounds stated within these classes are best possible.

THEOREM 6.1. *Let G satisfy $W_n(l_1,..., l_q; k_1,..., k_m)$ and $|V(G)| \geqslant q+m$. If G is not complete then G satisfies the lower bound (I) of Theorem* 4.1. *If G is not n-vertex connected then G satisfies exactly one of the following*:

(i) *$\kappa(G) \leqslant n-1$, and $|V(G)| \leqslant n + 2\max\{q-\alpha_1, m-\beta_1\} - 3$;*

(ii) *$\kappa(G) \leqslant n-1$; and $|V(G)| \geqslant n + 2\max\{q-\alpha_1, m-\beta_1\} - 2$; and $h(G) \leqslant \min\{q-\alpha_1-1, m-\beta_1-1\}$; and at most one part P with respect to any $S \in C(G)$ will have $|V(P)| \geqslant \min\{q-\alpha_1, m-\beta_1\}$.*

Proof. Let G be a complete graph which satisfies the hypothesis. Suppose that $\max\{q-\alpha_1, m-\beta_1\} \geqslant 2$. If complete graph G is not n-vertex connected then $|V(G)| \leqslant n < n + 2\max\{q-\alpha_1, m-\beta_1\} - 3$. Suppose that $\max\{q-\alpha_1, m-\beta_1\} \leqslant 1$. Then $\kappa(G) \geqslant n$ by Corollary 3.1.1.

Let G be an incomplete graph which satisfies the hypothesis. Suppose $\kappa(G) \leqslant n-1$ and $|V(G)| \geqslant n + 2\max\{q-\alpha_1, m-\beta_1\} - 2$. If any $S \in C(G)$ has $(V_1, V_2)_S$ such that $|V_1| \geqslant \min\{q-\alpha_1, m-\beta_1\}$ then $h(G) \geqslant \min\{q-\alpha_1, m-\beta_1\}$ and Theorem 3.1(ii) is contradicted. Hence $h(G) \leqslant$

$\min\{q - \alpha_1 - 1, m - \beta_1 - 1\}$. For these constraints, it is still possible for $|V_2|$ to be arbitrarily large.

Suppose some $C \in C(G)$ admits two distinct parts, P_1 and P_2, such that $|V(P_1)| \geqslant \max\{q - \alpha_1, m - \beta_1\}$ and $|V(P_2)| \geqslant \max\{q - \alpha_1, m - \beta_1\}$. Then $(V_1, V_2)_C$ has $|V_2| \geqslant |V_1| \geqslant \min\{q - \alpha_1, m - \beta_1\}$. It follows that $h(G) \geqslant |V_1| \geqslant \min\{q - \alpha_1, m - \beta_1\}$, which is impossible. Hence, at most one part P with respect to any $S \in C(G)$ has $|V(P)| \geqslant \min\{q - \alpha_1, m - \beta_1\}$.

Examples show that all bounds of Theorem 6.1 are best possible. Furthermore, let $n \geqslant 6$ and $W_n(l_1, ..., l_q; k_1, ..., k_m)$ be given such that $\min\{q - \alpha_1, m - \beta_1\} \geqslant 2$. Then there exists a graph G which satisfies $W_n(l_1, ..., l_q; k_1, ..., k_m)$ and $\kappa(G) = n - 1$.

In order to see that the bound for $|V(G)|$ in Theorem 6.1(i) and Theorem 3.1(ii) is best possible then consider the following result by Mesner and Watkins.

EXAMPLE 6.1 [4, Theorem 5]. Let n be a positive integer. Let $k_1, ..., k_m$ be a nonincreasing sequence of positive integers such that $\sum_{j=1}^{m} k_j = n$ and $\beta_1 \leqslant m - 3$. Then $G(m - \beta_1 - 1,\ n - 1,\ m - \beta_1 - 1)$ satisfies $W_n(n; k_1, ..., k_m)$ and $\kappa(G) = n - 1$. In Example 6.1, $m - \beta_1 \geqslant 3$, $|V(G)| = n + 2\ \max\{q - \alpha_1, m - \beta_1\} - 3$ and $h(G) = m - \beta_1 - 1 \geqslant 2 > 1 = q - \alpha_1$.

In order to see that the bounds of Theorem 6.1(ii) are best possible then consider Example 4.2. For $n = 2k$, $(k \geqslant 3)$, then $G(k - 1, 2k - 1, k)$ satisfies $W_{2k}, (2, ..., 2; 2, ..., 2)$ where $|V(G)| = 4k - 2 = n + 2\max\{q - \alpha_1, m - \beta_1\} - 2$ and $h(G) = k - 1 = \min\{q - \alpha_1 - 1, m - \beta_1 - 1\}$. For $n = 2k + 1$, $(k \geqslant 3)$, $G(k - 1, 2k, k)$ satisfies $W_{2k+1}(3, 2, ..., 2; 3, 2, ..., 2)$. In this case, $|V(G)| = 4k - 1 = n + 2\ \max\{q - \alpha_1,\ m - \beta_1\} - 2$ and $h(G) = k - 1 = \min\{q - \alpha_1 - 1, m - \beta_1 - 1\}$. Hence, the bounds of Theorem 6.1 (ii) are best possible for $n \geqslant 6$.

Let $n \geqslant 6$ and $W_n(l_1, ..., l_q; k_1, ..., k_m)$ be given. Some arbitrarily large graph G which belongs to the class of Theorem 6.1(ii) then also satisfies $W_n(l_1, ..., l_q; k_1, ..., k_m)$.

THEOREM 6.2 [2, Theorem 2.13]. *Let n and N be positive integers such that $n \geqslant 6$ and $N \geqslant 2n$. Let positive integer t and $W_n(l_1, ..., l_q; k_1, ..., k_m)$ be such that $2 \leqslant t + 1 \leqslant \min\{q - \alpha_1, m - \beta_1\}$. Then $G = G(t, n - 1, N)$ satisfies $W_n(l_1, ..., l_q; k_1, ..., k_m)$ where $\kappa(G) = n - 1$.*

Sketch of proof. The full proof involves elaborate notation for vertices in the counting argument. However, the technique is apparent from a brief sketch.

Consider $G = G(t, n - 1, N)$ where $V(G) = U_1 \cup U_2 \cup U_3$; $|U_1| = t$, $|U_2| = n - 1$, $|U_3| = N$. Section subgraph $G(U_2 \cup U_3) = K_{N+n-1}$, a complete graph, which is at least $2n$-vertex connected. If $(A \cup B) \subseteq U_2 \cup U_3$, then G satisfies $W_n(l_1, ..., l_q; k_1, ..., k_m)$ by Theorem 2.2.

Suppose $A \cap U_1 \neq \varnothing$ and $B \cap U_1 \neq \varnothing$. At least one multiple-arc end point of A and at least one multiple-arc end point of B must belong to $U_2 \cup U_3$ because $h(G) = |U_1| = t \leqslant \min\{q - \alpha_1 - 1, m - \beta_1 - 1\}$ by hypothesis. It follows that $A \cap U_1$ is a set of end points for at most $(n - 2)$ arcs of the openly disjoint family of AB arcs. Similarly, $B \cap U_1$ is a set of end points for at most $(n - 2)$ arcs of the same family.

If $A \cap U_2 = \varnothing$, then each element of $A \cap U_1$ may be linked, by single edges, to an appropriate number of distinct vertices of U_2. These single edges are the initial edges of the requisite number of AB-arcs from each arc end point of $A \cap U_1$. If $U_2 \cap A \neq \varnothing$, then every element of $A \cap U_2$ represents a reduction from n of at least one arc in the number of AB-arcs which exit from U_1 to U_2. Hence, there remain sufficient vertices of $U_2 - A$ to serve as the requisite number of end points of initial edges of AB-arcs with arc end points belonging to $A \cap U_1$. Let Q denote the subset of $U_2 - A$ which serves as the second vertices of AB-arcs with arc end points belonging to $A \cap U_1$.

Similarly, each element of $B \cap U_1$ may be linked by a single edge to an appropriate number of vertices of $U_2 - B$. These single edges serve in the requisite number of AB-arcs from each arc end point of $B \cap U_1$. Let this subset of $(U_2 - B)$ which are linked by a single edge to $(U_1 \cap B)$ be denoted by R.

If $Q \cap R \neq \varnothing$, then elements of $Q \cap R$ serve as intermediate vertices of 2-edge AB-arcs. Regardless, it only remains to show that an openly disjoint family of at most n arcs exists in $G(U_2 \cup U_3)$ between $(Q \cup A)$ and $(R \cup B)$. Such a family does exist by Theorem 2.2.

If either $A \cap U_1 = \varnothing$ or $B \cap U_1 = \varnothing$, then the previous argument prevails along a more simple course. Hence, G satisfies the given $W_n(l_1, ..., l_q; k_1, ..., k_m)$ but $\kappa(G) = n - 1$.

The converse of Theorem 6.1 is false. It is easy to find graphs of the form $G(\theta_1, \theta_2, \theta_3)$ which may satisfy either 6.1(i) or 6.1(ii), but the graph does not satisfy some corresponding $W_n(l_1, ..., l_q; k_1, ..., k_m)$.

In summary, a graph which satisfies $W_n(l_1, ..., l_q; k_1, ..., k_m)$ is classified in terms of certain bounds on vertex-connectedness, vertex sets, and the structure of parts with respect to minimum vertex cutsets. Extremal examples are given to show these bounds are best possible.

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